Bab 5: Transformasi Fourier

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Orthogonal Function

- Two functions are orthogonal in the interval [a, b] if

$$\int_a^b f_u(t) \cdot \bar{f}_v(t)\, dt = \begin{cases} 0 & \text{if } u \neq v \\ c & \text{if } u = v \end{cases}$$

where fu and fv are functions with real and imaginary components, f indicates complex conjugate of the function f, and c is any number different than zero.

- Harmonics fulfill the orthogonality property

$$\int_0^T \sin\left(\frac{2\pi}{T}t\right) \cdot \sin\left(u\frac{2\pi}{T}t\right) \cdot dt = \begin{cases} 0 & \text{if } u \neq 1 \\ \frac{T}{2} & \text{if } u = 1 \end{cases}$$

$$\int_0^T \cos\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right) \cdot dt = \begin{cases} 0 & \text{if } u \neq 1 \\ \frac{T}{2} & \text{if } u = 1 \end{cases}$$

$$\int_0^T \sin\left(\frac{2\pi}{T}t\right) \cdot \cos\left(u\frac{2\pi}{T}t\right) \cdot dt = 0 \qquad \text{for all } u$$

Fourier Series (deret Fourier)

- Sinyal periodik dapat dinyatakan dengan deret Fourier:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$$

ck adalah koefisien deret Fourier, $\Omega_0 = 2\pi/T_0$ adalah frekuensi dasar dan k Ω_0 adalah frekuensi harmonik ke k

- Koefisien Fourier dinyatakan dengan

$$c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\Omega_0 t}\, dt$$

- For an odd function *x(t), it is easier to calculate the interval from 0 to T0. For an even function, integration* from *−T0/2 to T0/2 is commonly used*

Contoh

- Deret pulsa kotak adalah periodik sinyal yang dapat dinyatakan dengan

$$x(t) = \begin{cases} A, & kT_0 - \tau/2 \leq t \leq kT_0 + \tau/2 \\ 0, & \text{otherwise} \end{cases}$$

Karena x(t) adalah fungsi genap, maka koefisien Fouriernya dapat dinyatakan dengan

$$c_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A e^{-jk\Omega_0 t}\, dt = \frac{A}{T_0}\left[\frac{e^{-jk\Omega_0 t}}{-jk\Omega_0}\bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}\right] = \frac{A\tau}{T_0}\,\frac{\sin(k\Omega_0\tau/2)}{k\Omega_0\tau/2}.$$

Nilai ck maksimum *Aτ/T0 pada frekuensi DC $\Omega_0 = 0$,*

Transformasi Fourier

- Secara praktis sinyal sering tidak periodik, sinyal periodik ini didekati dengan sinyal periodik yang periodanya tak hingga. i.e. $T_0 \rightarrow \infty$ *(or* $\Omega_0 \rightarrow 0$*)*

- Transformasi Fourier untuk sinyal analog

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,\mathrm{d}t \qquad \text{atau} \qquad X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,\mathrm{d}t.$$

- Transformasi baliknya

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}\,\mathrm{d}\Omega.$$

Contoh

- Hitung Transformasi Fourier dari

$$x(t) = e^{-at}u(t)$$

untuk a>0
Jawab:

$$X(\Omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\Omega t}\,\mathrm{d}t = \int_{0}^{\infty} e^{-(a+j\Omega)t}\,\mathrm{d}t$$

$$= \frac{1}{a+j\Omega}.$$

- Untuk fungsi x(t) yang ada hanya pada interval tertentu T_0 i.e. *x(t) = 0 for |t| > T0/2,* Koefisien deret Fouriernya dapat dinyatakan dengan

$$c_k = \frac{1}{T_0}X(k\Omega_0).$$

Discrete Fourier Transform

Discrete-Time Fourier Transform infinite duration
DTFT dari sinyal diskrit x(nT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

Discrete Fourier Transform (DFT)→ Finite Duration

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}, \qquad k = 0, 1, \ldots, N-1,$$

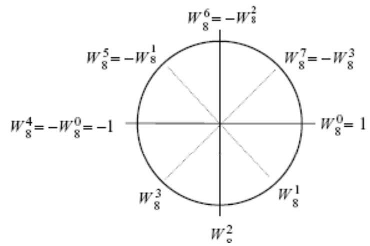
- **DFT dapat juga ditulis dengan**

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \qquad k = 0, 1, \ldots, N-1, \qquad \text{dengan}$$

$$W_N^{kn} = e^{-j\left(\frac{2\pi}{N}\right)kn} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right), \qquad 0 \le k, n \le N-1.$$

Twiddle factors for DFT, *N = 8*

$$W_N^{kn} = e^{-j\left(\frac{2\pi}{N}\right)kn} = \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right), \qquad 0 \le k, n \le N-1.$$



The parameter W_N^{kn} is called the twiddle factors of the DFT. Because $W_N^N = e^{-j2\pi} = 1 = W_N^0$, W_N^k, $k = 0, 1, \ldots, N-1$ are the N roots of unity in clockwise direction on the unit circle. It can be shown that $W_N^{N/2} = e^{-j\pi} = -1$.

The twiddle factors have the symmetry property

$$W_N^{k+N/2} = -W_N^k, \qquad 0 \le k \le N/2 - 1, \tag{6.16}$$

and the periodicity property

$$W_N^{k+N} = W_N^k. \tag{6.17}$$

Example 6.6: Consider the finite-length signal

$$x(n) = a^n, \qquad n = 0, 1, \ldots, N-1,$$

where $0 < a < 1$. The DFT of $x(n)$ is computed as

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j(2\pi k/N)n} = \sum_{n=0}^{N-1} \left(a e^{-j2\pi k/N}\right)^n$$

$$= \frac{1-\left(a e^{-j2\pi k/N}\right)^N}{1-a e^{-j2\pi k/N}} = \frac{1-a^N}{1-a e^{-j2\pi k/N}}, \quad k = 0, 1, \ldots, N-1.$$

Inverse Discrete Fourier Transform

- IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1.$$

DFT Matrix

The DFT and IDFT defined in Equations (6.14) and (6.18), respectively, can be expressed in matrix-vector form as

$$\mathbf{X} = \mathbf{W}\mathbf{x} \tag{6.19}$$

and

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^*\mathbf{X}, \tag{6.20}$$

where $\mathbf{x} = [x(0)\,x(1)\ldots x(N-1)]^T$ is the signal vector, the complex vector $\mathbf{X} = [X(0)X(1)\ldots X(N-1)]^T$ contains the DFT coefficients, and the NxN twiddle-factor matrix (or DFT matrix) $\mathbf{W}$ is defined by

$$\mathbf{W} = \left[W_N^{kn}\right]_{0\le k,n\le N-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)^2} \end{bmatrix}, \tag{6.21}$$

and $\mathbf{W}^*$ is the complex conjugate of the matrix $\mathbf{W}$. Since $\mathbf{W}$ is a symmetric matrix, the inverse matrix $\mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^*$ was used to derive Equation (6.20).

Example 6.7: Given $x(n) = \{1, 1, 0, 0\}$, the DFT of this 4-point sequence can be computed using the matrix formulation as

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix},$$

where we used symmetry and periodicity properties given in Equations (6.16) and (6.17) to obtain $W_4^0 = W_4^4 = 1$, $W_4^1 = W_4^9 = -j$, $W_4^2 = W_4^6 = -1$, and $W_4^3 = j$. The IDFT can be computed as

$$\mathbf{x} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ 1 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ 1 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} \mathbf{X} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Magnituda dan fasa dari DFT

- Spektrum Magnituda dari DFT

$$|X(k)| = \sqrt{\{\mathrm{Re}[X(k)]\}^2 + \{\mathrm{Im}[X(k)]\}^2}$$

- Spektrum Fasa dari DFT

$$\phi(k) = \tan^{-1}\left\{\frac{\mathrm{Im}[X(k)]}{\mathrm{Re}[X(k)]}\right\}$$

Important Properties

- Linearity: Jika {x(n)} dan {y(n)} adalah sinyal digital dengan panjang yang sama maka

$$\begin{aligned}\mathrm{DFT}[ax(n) + by(n)] &= a\mathrm{DFT}[x(n)] + b\mathrm{DFT}[y(n)]\\ &= aX(k) + bY(k),\end{aligned}$$

- Complex Conjugate
  Jika urutan (sequence) {x(n), 0 ≤ *n ≤ N − 1} bernilai real maka*

$$X(M + k) = X^*(M - k), \qquad 0 \le k \le M,$$

dengan *M = N/2 jika N adalah genap, atau M = (N − 1)/2 jika N adalah ganjil. Sifat ini menunjukkan hanya koefisien* (*M + 1) DFT pertama dari k = 0 to M yang independet*

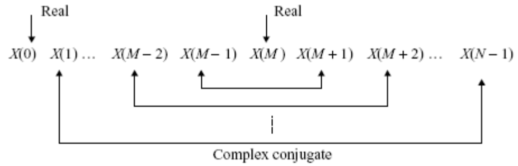


Figure 6.3 Complex-conjugate property, *N* is an even number

- Symmetry:
  |X(k)| = |X(N − k)|, k = 1, 2, . . . , M − 1
  φ(k) = −φ(N − k), k = 1, 2, . . . , M − 1.

- **DFT dan z-transform**

$$X(k) = X(z)|_{z=e^{j(\frac{2\pi}{N})k}}, \quad k = 0, 1, \ldots, N-1.$$

Fast Fourier Transforms

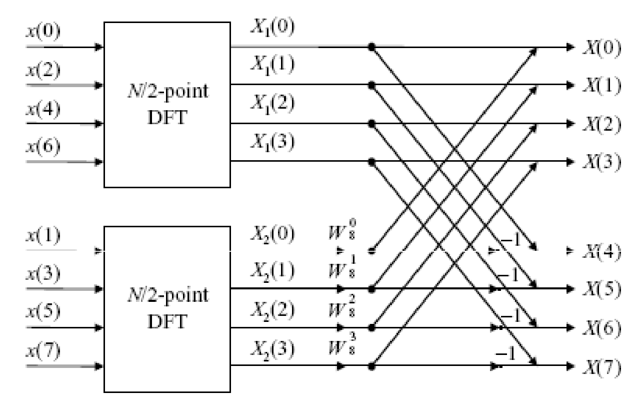
- Merupakan algoritma yang lebih efisien dari DFT
- DFT membutuhkan N^2 perkalian dan (N^2-N) penjumlahan, total operasi arithmetik 4N2.
- Redudansi DFT dapat dikurangi dengan memanfaatkan twiddle factor

$$W_N^{kn} = W_N^{(kn) \bmod N}, \qquad \text{for } kn > N$$

dan

$$W_N^N = 1$$

FFT dengan Butterflies Network



DECIMATION in TIME

- X(n) dibagi menjadi urutan genap

  $x_1(m) = x(2m), m = 0, 1, . . . , (N/2) − 1$

  dan ganjil

  $x_2(m) = x(2m + 1), m = 0, 1, . . . , (N/2) − 1$

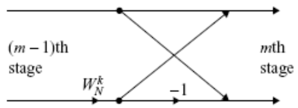


Figure 6.6 Flow graph for a butterfly computation

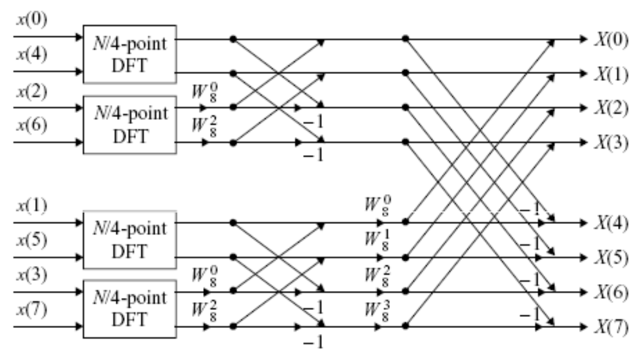


Figure 6.7 Flow graph illustrating second step of N-point DFT, $N = 8$

TRUNCATION, LEAKAGE, AND WINDOWS

Short duration transients can be adequately recorded from beginning to end. Some A/D converters even permit pretriggering to gather the background signal prior to the transient. However, long-duration or ongoing signals are inevitably truncated and we only see a finite "window of the signal".

The Hanning and Hamming windows are two common windowing functions:

$$\textit{Hanning} \quad w_i = \begin{cases} \frac{1}{2} + \frac{1}{2} \cdot \cos\left[\frac{2\pi}{E}(i - M)\right] & |i - M| \le \frac{E}{2} \\ 0 & \text{otherwise} \end{cases} \tag{5.33}$$

$$\textit{Hamming} \quad w_i = \begin{cases} 0.54 + 0.46 \cdot \cos\left[\frac{2\pi}{E}(i - M)\right] & |i - M| \le \frac{E}{2} \\ 0 & \text{otherwise} \end{cases} \tag{5.34}$$

These windows are centered around $i = M$ and have a time width $E \cdot \Delta t$. In this format, the rectangular window becomes

$$\textit{Rectangular} \quad w_i = \begin{cases} 1 & |i - M| \le \frac{E}{2} \\ 0 & \text{otherwise} \end{cases} \tag{5.35}$$

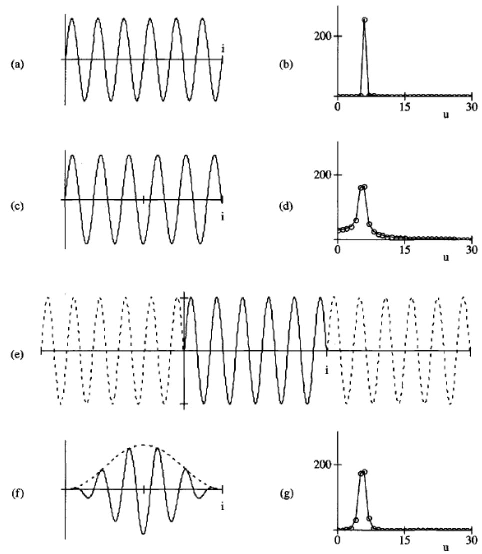


Figure 5.6 Truncation and windowing: (a, b) the DFT of a single frequency sinusoid is an impulse if it completes an integer number of cycles in the duration of the signal T; (c, d) this signal has an incomplete number of cycles; its DFT is not an impulse and has a static component; (e) periodic assumption in the DFT; (f) signal in frame 'c' but windowed with smooth transition towards zero ends; (g) autospectrum of the windowed signal

PADDING

A longer duration N • At signal renders a better frequency resolution Af = 1/(N • At). Therefore, a frequently used technique to enhance the frequency resolution of a stored signal length N consists of "extending" the signal by appending values to a length M > N. This approach requires careful consideration.

There are various "signal extension" strategies. Zero padding, the most common extension strategy, consists of appending zeros to the signal. Constant padding extends the signal by repeating the last value. Linear padding extends the signal while maintaining the first derivative at the end of the signal constant. Finally, periodic padding uses the same signature for padding.

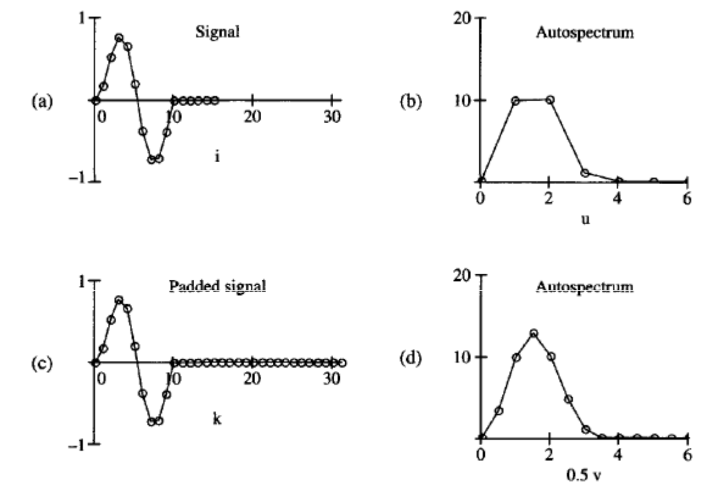


Figure 5.7 Time and frequency resolution: (a, b) original N = 16 signal and its auto spectrum; (c, d) zero-padded signal with N = 32 and its auto spectrum. Padding increases frequency resolution. The peak in the autospectral density of the original signal is absent because there is no corresponding harmonic. (Note: the time interval Δt is kept constant, the number of points N is doubled, and the frequency interval is halved.)

THE TWO-DIMENSIONAL DISCRETE FOURIER TRANSFORM