Chap 8: Differential Solution using Laplace Transform

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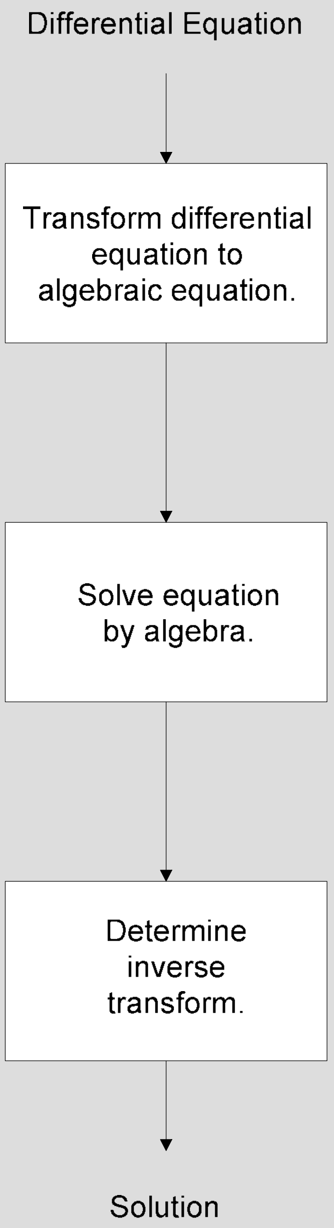
Differential Equations with Laplace Transform Methods

The Pierre Simon, marquis de Laplace transform was developed by the French mathematician by the same name (1749-1827) and was widely adapted to engineering problems in the last century. Its utility lies in the ability to convert differential equations to algebraic forms that are more easily solved. The notation has become very common in certain areas as a form of engineering “language” for dealing with systems.

Figure 8-1. Steps involved in using the Laplace transform.



Laplace Transformation

$$L[f(t)] = F(s)$$

$$L^{-1}[F(s)] = f(t)$$

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

Basic Theorems of Linearity

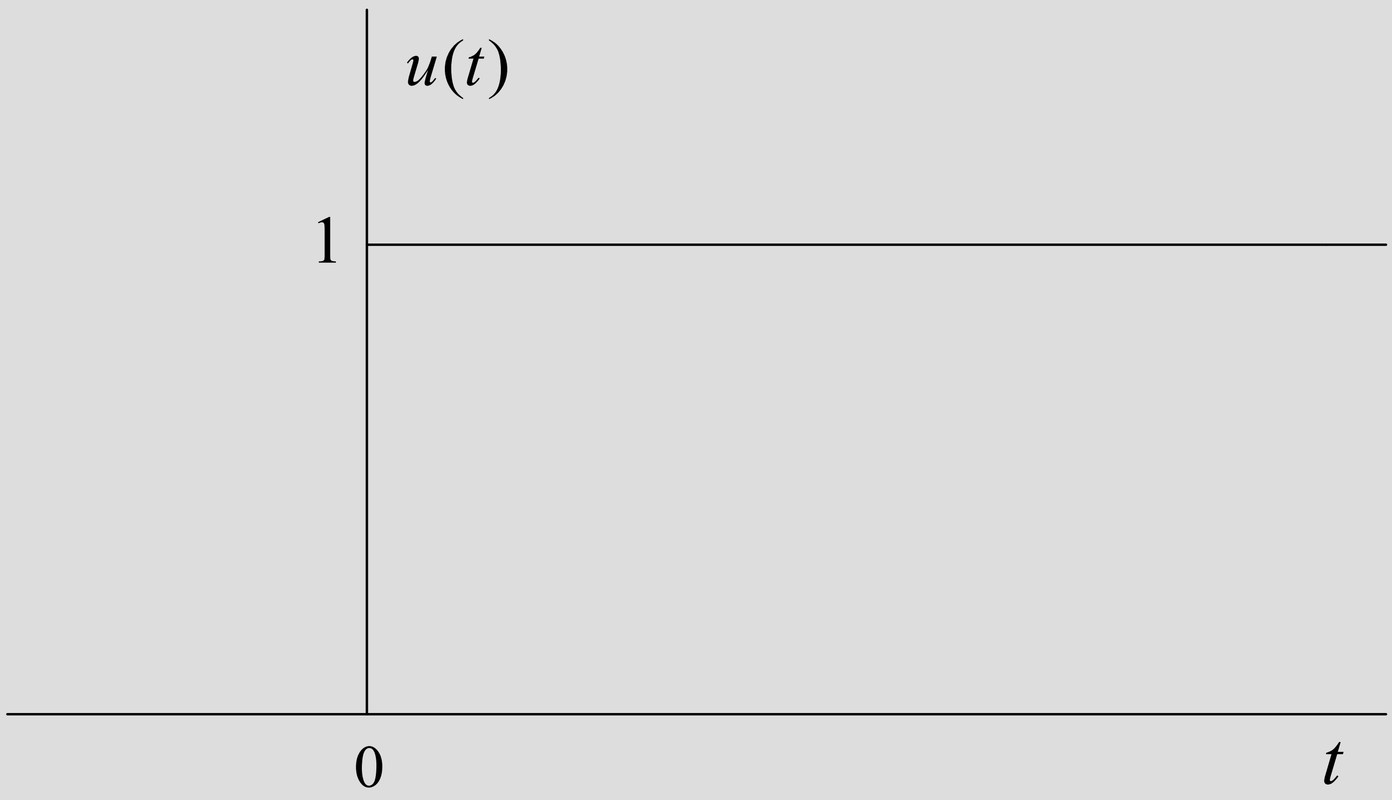
$$L[Kf(t)] = KL[f(t)] = KF(s)$$

$$L[f_1(t) + f_2(t)] = L[f_1(t)] + L[f_2(t)]$$
$$= F_1(s) + F_2(s)$$

The Laplace transform of a product is **not** the product of the transforms.

$$L[f_1(t) f_2(t)] \neq F_1(s) F_2(s)$$

Figure 8-2. Illustration of the unit step function.



Example 8-1. Derive the Laplace transform of the unit step function.

$$F(s) = \int_0^\infty (1)e^{-st}\,dt$$

$$F(s) = \left.\frac{e^{-st}}{-s}\right]_0^\infty = 0 - \left(\frac{e^{-0}}{-s}\right) = \frac{1}{s}$$

Example 8-2. Derive the Laplace transform of the exponential function

$$f(t) = e^{-\alpha t}$$

$$F(s) = \int_0^\infty e^{-\alpha t} e^{-st} dt = \int_0^\infty e^{-(\alpha + s)t} dt$$

$$= \left. \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right]_0^\infty = 0 - \frac{e^{-0}}{-(s+\alpha)}$$

$$= \frac{1}{s+\alpha}$$

Table 8-1. Common transform pairs.

$f(t)$	$F(s) = L[f(t)]$	
1 or $u(t)$	$\frac{1}{s}$	T-1
$e^{-\alpha t}$	$\frac{1}{s+\alpha}$	T-2
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	T-3
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	T-4
$e^{-\alpha t} \sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	T-5*
$e^{-\alpha t} \cos \omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	T-6*
t	$\frac{1}{s^2}$	T-7
t^n	$\frac{n!}{s^{n+1}}$	T-8
$e^{-\alpha t} t^n$	$\frac{n!}{(s+\alpha)^{n+1}}$	T-9
$\delta(t)$	1	T-10

*Use when roots are complex.

Example 8-3. A force in newtons (N) is given below. Determine the Laplace transform.

$$f(t) = 50u(t)$$

$$F(s) = \frac{50}{s}$$

Example 8-4. A voltage in volts (V) starting at t = 0 is given below. Determine the Laplace transform.

$$v(t) = 5e^{-2t}\sin 4t$$

$$V(s) = L[v(t)] = 5\cdot\frac{4}{(s+2)^2+(4)^2}$$

$$= \frac{20}{s^2+4s+4+16} = \frac{20}{s^2+4s+20}$$

Example 8-5. A pressure in pascals (p) starting at t = 0 is given below. Determine the Laplace transform.

$$p(t) = 5\cos 2t + 3e^{-4t}$$

$$P(s) = L[p(t)] = 5 \cdot \frac{s}{s^2 + (2)^2} + 3 \cdot \frac{1}{s+4}$$

$$= \frac{5s}{s^2+4} + \frac{3}{s+4}$$

Inverse Laplace Transforms by Identification

When a differential equation is solved by Laplace transforms, the solution is obtained as a function of the variable s. The inverse transform must be formed in order to determine the time response. The simplest forms are those that can be recognized within the tables and a few of those will now be considered.

Example 8-6. Determine the inverse transform of the function below.

$$F(s) = \frac{5}{s} + \frac{12}{s^2} + \frac{8}{s+3}$$

$$f(t) = 5 + 12t + 8e^{-3t}$$

Example 8-7. Determine the inverse transform of the function below.

$$V(s) = \frac{200}{s^2 + 100}$$

$$V(s) = 20\left(\frac{10}{s^2 + (10)^2}\right)$$

$$v(t) = 20\sin 10t$$

Example 8-8. Determine the inverse transform of the function below.

$$V(s) = \frac{8s+4}{s^2+6s+13}$$

When the denominator contains a quadratic, check the roots. If they are real, a partial fraction expansion will be required. If they are complex, the table may be used. In this case, the roots are

$$s_{1,2} = -3 \pm 2i$$

Example 8-8. Continuation.

$$s^2 + 6s + 13$$

$$= s^2 + 6s + (3)^2 + 13 - (3)^2$$

$$= s^2 + 6s + 9 + 4$$

$$= (s+3)^2 + (2)^2$$

Example 8-8. Continuation.

$$V(s) = \frac{8(s+3)}{(s+3)^2+(2)^2} + \frac{4-24}{(s+3)^2+(2)^2}$$

$$= \frac{8(s+3)}{(s+3)^2+(2)^2} - \frac{10(2)}{(s+3)^2+(2)^2}$$

$$v(t) = 8e^{-3t}\cos 2t - 10e^{-3t}\sin 2t$$

Forms for CCLODE constant (coefficient linear ordinary differential equations) Transforms

$$F(s) = \frac{N(s)}{D(s)}$$

$$N(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0$$

$$D(s) = b_m s^m + b_{m-1} s^{m-1} + ... + b_1 s + b_0$$

$$F(s) = \frac{N(s)}{b_m (s - p_1)(s - p_2)....(s - p_m)}$$

The roots of $D(s)$ are called *poles* and they may be classified in four ways.

1. Real poles of first order.
2. Complex poles of first order (including purely imaginary poles)
3. Real poles of multiple order
4. Complex poles of multiple order (including purely imaginary poles)

Partial Fraction Expansion
Real Poles of First Order

$$F(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \dots + \frac{A_r}{s - p_r} + R(s)$$

$$A_k = (s - p_k)F(s)\Big]_{s = p_k}$$

$$f_1(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \dots A_r e^{p_r t}$$

Example 8-9. Determine inverse transform of function below.

$$F(s) = \frac{s+6}{s^2+3s+2} = \frac{s+6}{(s+1)(s+2)}$$

$$F(s) = \frac{s+6}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

$$A_1 = (s+1)F(s)\Big]_{s=-1} = \frac{s+6}{s+2}\Bigg]_{s=-1} = \frac{-1+6}{-1+2} = 5$$

Example 8-9. Continuation.

$$A_2 = (s+2)F(s)\Big]_{s=-2} = \left.\frac{s+6}{s+1}\right]_{s=-2} = \frac{-2+6}{-2+1} = -4$$

$$F(s) = \frac{5}{s+1} - \frac{4}{s+2}$$

$$f(t) = 5e^{-t} - 4e^{-2t}$$

Example 8-10. Determine exponential portion of inverse transform of function below.

$$F(s) = \frac{50(s+3)}{(s+1)(s+2)(s^2+2s+5)}$$

$$F_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Example 8-10. Continuation.

$$A_1 = \left.\frac{50(s+3)}{(s+2)(s^2+2s+5)}\right]_{s=-1} = \frac{(50)(2)}{(1)(4)} = 25$$

$$A_2 = \left.\frac{50(s+3)}{(s+1)(s^2+2s+5)}\right]_{s=-2} = \frac{(50)(1)}{(-1)(5)} = -10$$

$$f_1(t) = 25e^{-t} - 10e^{-2t}$$

Partial Fraction Expansion for First-Order Complex Poles

$$(s^2 + bs + c) \Rightarrow p_{1,2} = -\alpha \pm i\omega$$

$$F(s) = \frac{As + B}{s^2 + bs + c} + R(s)$$

Example 8-11. Complete the inverse transform of Example 10-10.

$$\frac{50(s+3)}{(s+1)(s+2)(s^2+2s+5)} = \frac{25}{s+1} - \frac{10}{s+2} + \frac{As+B}{s^2+2s+5}$$

$$\frac{50(3)}{(1)(2)(5)} = \frac{25}{1} - \frac{10}{2} + \frac{B}{5} \qquad B = -25$$

$$\frac{50(4)}{(2)(3)(8)} = \frac{25}{2} - \frac{10}{3} + \frac{A+B}{8} \qquad A = -15$$

$$F(s) = \frac{25}{s+1} - \frac{10}{s+2} + \frac{-15s-25}{s^2+2s+5}$$

Example 8-11. Continuation.

$$F_2(s) = \frac{-15s - 25}{s^2 + 2s + 5}$$

$$s^2 + 2s + 5 = s^2 + 2s + 1 + 5 - 1 = (s+1)^2 + (2)^2$$

$$F_2(s) = \frac{-15s - 25}{(s+1)^2 + (2)^2} = \frac{-15(s+1)}{(s+1)^2 + (2)^2} + \frac{-5(2)}{(s+1)^2 + (2)^2}$$

$$f(t) = f_1(t) + f_2(t)$$

$$= 25e^{-t} - 10e^{-2t} - 15e^{-t}\cos 2t - 5e^{-t}\sin 2t$$

Second-Order Real Poles

Assume that $F(s)$ contains a denominator factor of the form $(s+\alpha)^2$. The expansion will take the form shown below.

$$F(s) = \frac{C_1}{(s+\alpha)^2} + \frac{C_2}{s+\alpha} + R(s)$$

$$C_1 = (s+\alpha)^2 F(s) \Big]_{s=-\alpha}$$

$$f_1(t) = C_1 t e^{-\alpha t} + C_2 e^{-\alpha t} = (C_1 t + C_2) e^{-\alpha t}$$

Example 8-12. Determine inverse transform of function below.

$$F(s) = \frac{60}{s(s+2)^2}$$

$$F(s) = \frac{60}{s(s+2)^2} = \frac{A}{s} + \frac{C_1}{(s+2)^2} + \frac{C_2}{(s+2)}$$

$$A = sF(s)\Big]_{s=0} = \frac{60}{(s+2)^2}\Bigg]_{s=0} = \frac{60}{(0+2)^2} = 15$$

$$C_1 = (s+2)^2 F(s)\Big]_{s=-2} = \frac{60}{s}\Bigg]_{s=-2} = \frac{60}{-2} = -30$$

Example 8-12. Continuation.

$$F(s)=\frac{60}{s(s+2)^2}=\frac{15}{s}-\frac{30}{(s+2)^2}+\frac{C_2}{s+2}$$

$$\frac{60}{(1)(1+2)^2}=\frac{15}{1}-\frac{30}{(1+2)^2}+\frac{C_2}{(1+2)} \qquad C_2=-15$$

$$F(s)=\frac{60}{s(s+2)^2}=\frac{15}{s}-\frac{30}{(s+2)^2}-\frac{15}{s+2}$$

$$f(t)=15-30te^{-2t}-15e^{-2t}=15-15e^{-2t}(1+2t)$$

Laplace Transform Operations

$f(t)$	$F(s)$	
$f'(t)$	$sF(s) - f(0)$	O-1
$\int_0^t f(t)dt$	$\frac{F(s)}{s}$	O-2
$e^{-\alpha t} f(t)$	$F(s+\alpha)$	O-3
$f(t-T)u(t-T)$	$e^{-sT} F(s)$	O-4
$f(0)$	$\lim_{s\to\infty} sF(s)$	O-5
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)^*$	O-6

*Poles of $sF(s)$ must have negative real parts.

Significant Operations for Solving Differential Equations

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$$L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

Procedure for Solving DEs

$$b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y = f(t)$$

$$L\left[b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y \right] = L\left[f(t) \right]$$

$$b_2 \left[s^2 Y(s) - sy(0) - y'(0) \right]$$

$$+ b_1 \left[sY(s) - y(0) \right] + b_0 Y(s) = F(s)$$

$$Y(s) = \frac{F(s)}{b_2 s^2 + b_1 s + b_0} + \frac{s b_2 y(0) + b_2 y'(0) + b_1 y(0)}{b_2 s^2 + b_1 s + b_0}$$

Example 8-13. Solve DE shown below.

$$\frac{dy}{dt} + 2y = 12 \qquad y(0) = 10$$

$$L\left[\frac{dy}{dt}\right] + 2L[y] = L[12]$$

$$sY(s) - 10 + 2Y(s) = \frac{12}{s}$$

$$(s+2)Y(s) = 10 + \frac{12}{s}$$

$$Y(s) = \frac{10}{s+2} + \frac{12}{s(s+2)}$$

Example 8-13. Continuation.

$$\frac{12}{s(s+2)} = \frac{A_1}{s} + \frac{A_2}{s+2}$$

$$A_1 = s\left[\frac{12}{s(s+2)}\right]_{s=0} = \left[\frac{12}{s+2}\right]_{s=0} = 6$$

$$A_2 = (s+2)\left[\frac{12}{s(s+2)}\right]_{s=-2} = \left[\frac{12}{s}\right]_{s=-2} = -6$$

$$Y(s) = \frac{10}{s+2} + \frac{6}{s} - \frac{6}{s+2} = \frac{6}{s} + \frac{4}{s+2}$$

$$y(t) = 6 + 4e^{-2t}$$

Example 8-14. Solve DE shown below.

$$\frac{dy}{dt} + 2y = 12\sin 4t \qquad y(0) = 10$$

$$sY(s) - 10 + 2Y(s) = \frac{12(4)}{s^2 + 16}$$

$$Y(s) = \frac{10}{s+2} + \frac{48}{(s+2)(s^2+16)}$$

$$\frac{48}{(s+2)(s^2+16)} = \frac{A}{s+2} + \frac{B_1 s + B_2}{s^2+16}$$

Example 8-14. Continuation.

$$A = \left.\frac{48}{s^2+16}\right]_{s=-2} = \frac{48}{20} = 2.4$$

$$\frac{48}{(s+2)(s^2+16)} = \frac{2.4}{s+2} + \frac{B_1 s + B_2}{s^2+16}$$

$$\frac{48}{(2)(16)} = \frac{2.4}{2} + \frac{B_2}{16} \qquad B_2 = 4.8$$

$$\frac{48}{(1)(17)} = \frac{2.4}{1} + \frac{-B_1 + B_2}{17} \qquad B_1 = -2.4$$

Example 8-14. Continuation.

$$Y(s) = \frac{10}{s+2} + \frac{2.4}{s+2} - \frac{2.4s}{s^2+16} + \frac{4.8}{s^2+16}$$

$$y(t) = 12.4e^{-2t} - 2.4\cos 4t + 1.2\sin 4t$$

Example 8-15. Solve DE shown below.

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=24$$

$$y(0)=10 \text{ and } y'(0)=0$$

$$s^2Y(s)-10s-0+3\left[sY(s)-10\right]+2Y(s)=\frac{24}{s}$$

$$Y(s)=\frac{24}{s(s^2+3s+2)}+\frac{10s+30}{s^2+3s+2}$$

$$=\frac{24}{s(s+1)(s+2)}+\frac{10s+30}{(s+1)(s+2)}$$

Example 8-15. Continuation.

$$\frac{24}{s(s+1)(s+2)} = \frac{12}{s} - \frac{24}{s+1} + \frac{12}{s+2}$$

$$\frac{10s+30}{(s+1)(s+2)} = \frac{20}{s+1} - \frac{10}{s+2}$$

$$F(s) = \frac{12}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

$$f(t) = 12 - 4e^{-t} + 2e^{-2t}$$

Example 8-16. Solve DE shown below.

$$\frac{d^2 y}{dt^2}+2\frac{dy}{dt}+5y=20$$

$$y(0)=0 \text{ and } y'(0)=10$$

$$s^2Y(s)-0-10+2\left[sY(s)-0\right]+5Y(s)=\frac{20}{s}$$

$$Y(s)=\frac{20}{s(s^2+2s+5)}+\frac{10}{s^2+2s+5}$$

Example 8-16. Continuation.

$$\frac{20}{s(s^2+2s+5)} = \frac{4}{s} + \frac{As+B}{(s^2+2s+5)}$$

$$\frac{20}{(1)(1+2+5)} = \frac{4}{1} + \frac{A+B}{(1+2+5)}$$

$$\frac{20}{(-1)(1-2+5)} = \frac{4}{-1} + \frac{-A+B}{(1-2+5)}$$

$$A = -4 \qquad B = -8$$

Example 8-16. Continuation.

$$Y(s) = \frac{4}{s} + \frac{-4s-8}{s^2+2s+5} + \frac{10}{s^2+2s+5} = \frac{4}{s} + \frac{-4s+2}{s^2+2s+5}$$

$$s^2+2s+5 = s^2+2s+1+5-1 = (s+1)^2+(2)^2$$

$$Y(s) = \frac{4}{s} + \frac{-4(s+1)}{(s+1)^2+(2)^2} + \frac{3(2)}{(s+1)^2+(2)^2}$$

$$y(t) = 4 - 4e^{-t}\cos 2t + 3e^{-t}\sin 2t$$