Chap 10a: Example of Continuous Fourier Transform

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Chap 10a: Fourier Series and Fourier Transform

3. **Basis functions** (3 lectures): Concept of basis function. **Fourier series representation of time functions. Fourier transform** and its properties. Examples, transform of simple time functions.

Specific objectives for today:

- Examples of **Fourier series** of periodic functions
- Rational and definition of **Fourier transform**
- Examples of **Fourier transforms**

Note

- Note that in this lecture, we're initially looking at periodic signals which have a **Fourier series** representation: a discrete set of complex coefficients
- However, we'll generalise this to non-periodic signals that which have a **Fourier transform** representation: a complex valued function
- Fourier series **sum** becomes a Fourier transform **integral**

Example 1: Fourier Series sin($\omega_0 t$)

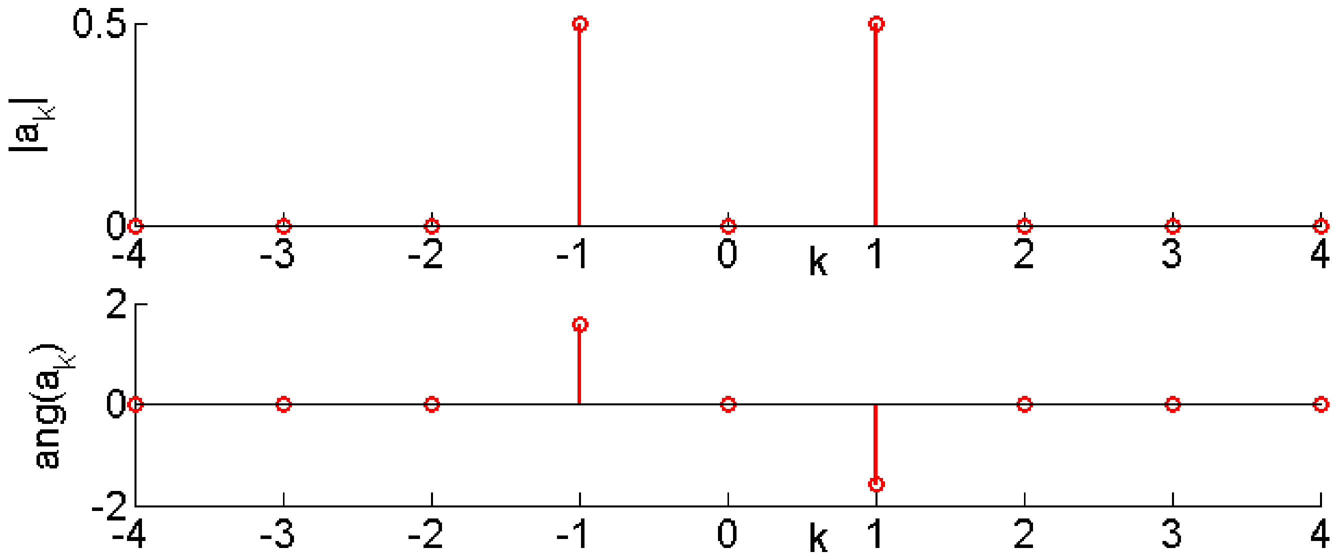
The fundamental period of sin($\omega_0 t$) is ω_0

By inspection we can write:

$$\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

So a_1 = 1/2j, a_{-1} = -1/2j and a_k = 0 otherwise

The magnitude and angle of the Fourier coefficients are:



Example 1a: Fourier Series $\sin(\omega_0 t)$

The Fourier coefficients can also be explicitly evaluated

$$a_0 = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \sin(\omega_0 t) dt = -\cos(\omega_0 t)\Big|_0^{2\pi/\omega_0} = 1 - 1 = 0$$

$$a_k = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \sin(\omega_0 t) e^{-jk\omega_0 t} dt = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \left(\tfrac{1}{2j} e^{j\omega_0 t} - \tfrac{1}{2j} e^{-j\omega_0 t} \right) e^{-jk\omega_0 t} dt$$

$$= \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \left(\tfrac{1}{2j} e^{j\omega_0 t} - \tfrac{1}{2j} e^{-j\omega_0 t} \right) e^{-jk\omega_0 t} dt$$

$$= \frac{\omega_0}{4\pi j} \int_0^{2\pi/\omega_0} e^{-j(k-1)\omega_0 t} - e^{-j(k+1)\omega_0 t} dt$$

When k = +1 or –1, the integrals evaluate to T and $-T$, respectively. Otherwise the coefficients are zero.

Therefore $a_1 = 1/2j$, $a_{-1} = -1/2j$

Example 2: Additive Sinusoids

Consider the additive sinusoidal series which has a fundamental frequency ω_0:

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \tfrac{\pi}{4}\right)$$

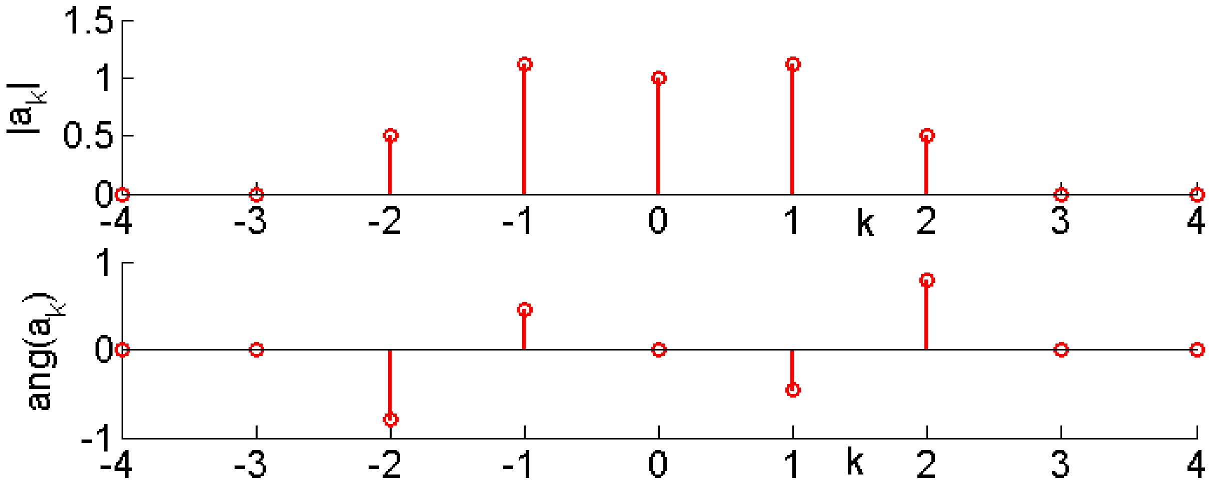
Again, the signal can be directly written as:

$$x(t) = 1 + \tfrac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) + (e^{j\omega_0 t} + e^{-j\omega_0 t}) + \tfrac{1}{2}(e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})})$$

$$= 1 + (1 + \tfrac{1}{2j})e^{j\omega_0 t} + (1 - \tfrac{1}{2j})e^{-j\omega_0 t} + \tfrac{1}{2}e^{j\frac{\pi}{4}}e^{j2\omega_0 t} + \tfrac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2\omega_0 t}$$

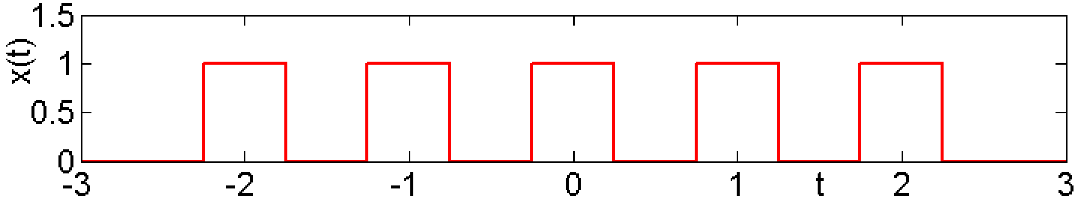
$$a_0 = 1 \quad a_1 = (1 - \tfrac{1}{2}j) \quad a_{-1} = (1 + \tfrac{1}{2}j) \quad a_2 = \tfrac{1}{2}e^{j\frac{\pi}{4}} \quad a_{-2} = \tfrac{1}{2}e^{-j\frac{\pi}{4}}$$

The Fourier series coefficients can then be visualised as:



Example 3: Periodic Step Signal

Consider the periodic square wave, illustrated by:



and is defined over one period as:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

Fourier coefficients:

$$a_0 = \tfrac{1}{T}\int_{-T_1}^{T_1} 1dt = \frac{2T_1}{T}$$

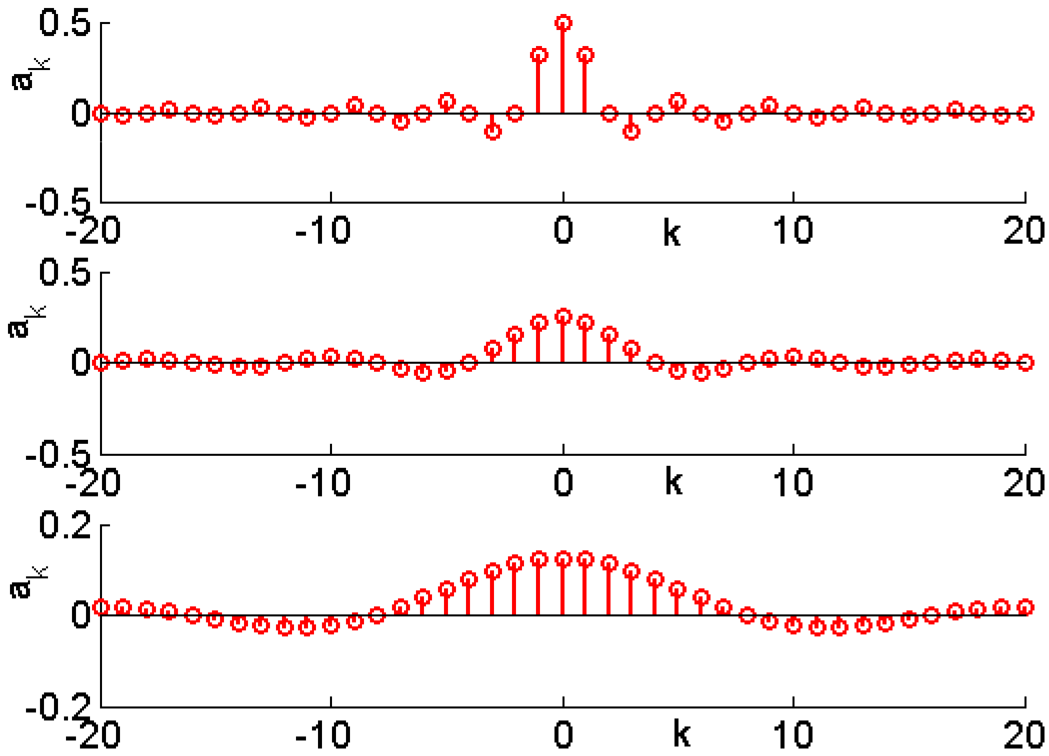
$$\begin{aligned} a_k &= \tfrac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = -\tfrac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} \\ &= \frac{2}{k\omega_0 T}\left(\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right) \\ &= 2\sin(k\omega_0 T_1)/k\omega_0 T \\ &= \sin(k\omega_0 T_1)/k\pi \end{aligned}$$

NB, these coefficients are real

Example 3a: Periodic Step Signal

Instead of plotting both the magnitude and the angle of the complex coefficients, we only need to plot the value of the coefficients.

Note we have an infinite series of non-zero coefficients



$T=4T_1$

$T=8T_1$

$T=16T_1$

Convergence of Fourier Series

Not every periodic signal can be represented as an infinite Fourier series, however just about all interesting signals can be (note that the step signal is discontinuous)

The Dirichlet conditions are necessary and sufficient conditions on the signal.

Condition 1. Over any period, *x*(*t*) must be absolutely integrable

$$\int_T |x(t)|dt < \infty$$

Condition 2. In any finite interval, *x*(*t*) is of bounded variation; that is there is no more than a finite number of maxima and minima during any single period of the signal

Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Further, each of these discontinuities are finite.

Fourier Series to Fourier Transform

For periodic signals, we can represent them as linear combinations of harmonically related complex exponentials

To extend this to non-periodic signals, we need to consider aperiodic signals as periodic signals with infinite period.

As the period becomes infinite, the corresponding frequency components form a continuum and the Fourier series sum becomes an integral (like the derivation of CT convolution)

Instead of looking at the coefficients a harmonically – related Fourier series, we'll now look at the **Fourier transform** which is a **complex valued function** in the frequency domain

Definition of the Fourier Transform

We will be referring to functions of time and their Fourier transforms. A signal $x(t)$ and its Fourier transform $X(j\omega)$ are related by the **Fourier transform synthesis** and **analysis** equations

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = F\{x(t)\}$$

and

$$x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = F^{-1}\{X(j\omega)\}$$

We will refer to $x(t)$ and $X(j\omega)$ as a **Fourier transform pair** with the notation

$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

As previously mentioned, the transform function $X()$ can roughly be thought of as a continuum of the previous coefficients

A similar set of Dirichlet convergence conditions exist for the Fourier transform, as for the Fourier series (T=(- ∞,∞))

Example 1: Decaying Exponential

Consider the (non-periodic) signal

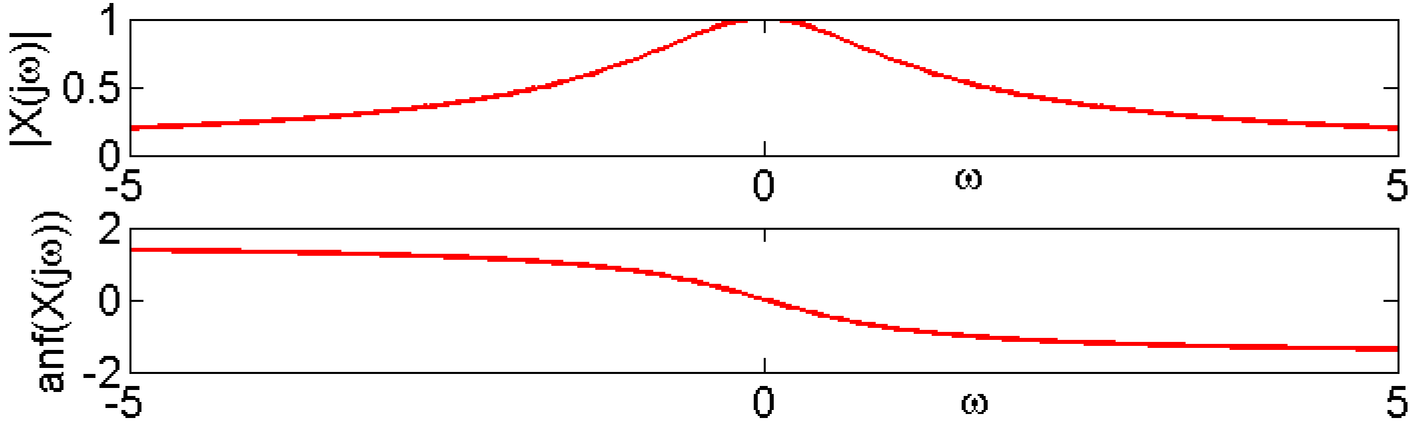
$$x(t) = e^{-at}u(t) \qquad a > 0$$

Then the Fourier transform is:

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{(a+j\omega)}$$



$a = 1$

Example 2: Single Rectangular Pulse

Consider the non-periodic rectangular pulse at zero

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| \geq T_1 \end{cases}$$

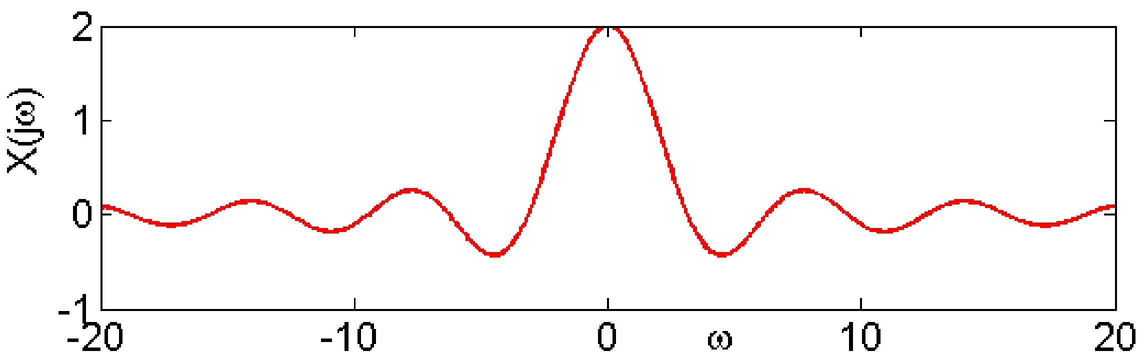
The Fourier transform is:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1}$$

$$= \frac{2\sin(\omega T_1)}{\omega}$$

Note, the values are real



$T_1 = 1$

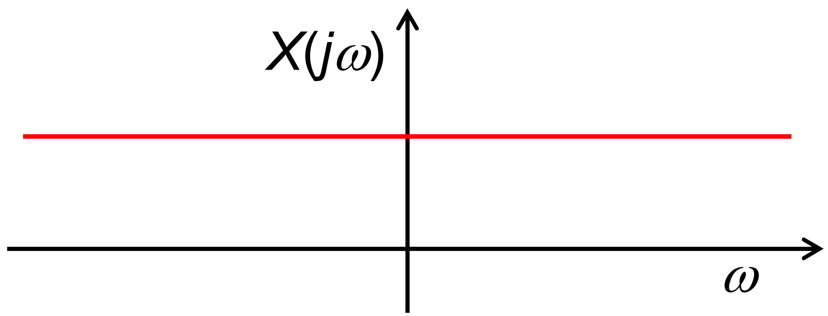
Example 3: Impulse Signal

The Fourier transform of the impulse signal can be calculated as follows:

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Therefore, the Fourier transform of the impulse function has a constant contribution for **all frequencies**



Example 4: Periodic Signals

A periodic signal violates condition 1 of the Dirichlet conditions for the Fourier transform to exist

However, lets consider a Fourier transform which is a single impulse of area 2π at a particular (harmonic) frequency $\omega=\omega_0$.

$$X(j\omega) = 2\pi\delta(\omega-\omega_0)$$

The corresponding signal can be obtained by:

$$x(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

which is a (complex) sinusoidal signal of frequency ω_0. More generally, when

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$$

Then the corresponding (periodic) signal is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

The Fourier transform of a periodic signal is a train of impulses at the harmonic frequencies with amplitude $2\pi a_k$

Chap 10a: Summary

Fourier series and **Fourier transform** is used to represent **periodic** and **non-periodic** signals in the **frequency domain**, respectively.

$$a_k = \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Looking at signals in the Fourier domain allows us to understand the frequency response of a system and also to design systems with a particular frequency response, such as filtering out high frequency signals.

You’ll need to complete the exercises to work out how to calculate the Fourier transform (and its inverse) and evaluate the frequency content of a signal

With Matlab

Matlab

To use the CT Fourier transform, you need to have the symbolic toolbox for Matlab installed. If this is so, try typing:

```
>> syms t;
>> fourier(cos(t))
>> fourier(cos(2*t))
>> fourier(sin(t))
>> fourier(exp(-t^2))
```

Note also that the ifourier() function exists so…

```
>> ifourier(fourier(cos(t)))
```